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AXISYMMETRICAL CONTACT PROBLEMS FOR PRESTRESSED DEFORMABLE BODIES

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Axisymmetrical contact problems are considered for a half-space and a layer of finite thickness h lying without friction on a deformable base previously stressed by uniform forces with intensity p applied at infinity. It is assumed that the material of deformable bodies is described by equations of physical nonlinear elasticity theory. The initial stress-strained state of the bodies (prestressing regime) is determined as an accurate solution of these equations. Action of a load on the surface of a layer (half-space) is considered as a small disturbance of the basic nonlinear stress field caused by prior loading. This makes it possible to perform linearization of all equations with respect to additional stresses, strains, and displacements. Contact problems for impression of a rigid stamp into a physically nonlinear material are posed for the linear equations obtained which are then reduced to first-order integral equations with a symmetrical irregular kernel with respect to distribution functions for contact pressures. Solutions of these equations are built up by means of asymptotic methods. Cases of loss of stability and deformability of a medium as a result of prestressing are studied. The effect of prestressing regime on the magnitude of contact pressures is studied.

1. Resolution equations for physically nonlinear (geometrically linear) elasticity theory for the case of axial symmetry and with the condition of absence of mass forces may be written as follows [1]:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_r}{\partial z} + \frac{\tau_{rz}}{r} = 0; \tag{1.1}$$

$$\begin{gathered} \varepsilon_r = \psi\sigma_r + (\varphi - \psi)\sigma, \ \varepsilon_\varphi = \psi\sigma_\varphi + (\varphi - \psi)\sigma, \\ \varepsilon_z = \psi\sigma_z + (\varphi - \psi)\sigma, \quad \varepsilon_{rz} = \psi\tau_{rz}, \quad \varepsilon = \varphi\sigma, \quad \gamma = 2\psi\tau, \\ \sigma = (\sigma_r + \sigma_\varphi + \sigma_z)/3, \quad \varepsilon = (\varepsilon_\varphi + \varepsilon_r + \varepsilon_z)/3, \\ \tau = \frac{1}{\sqrt{6}}\left[(\sigma_r - \sigma_\varphi)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\varphi - \sigma_z)^2 + 6\tau_{rz}^2\right]^{1/2}, \\ \gamma = \sqrt{\frac{2}{3}}\left[(\varepsilon_r - \varepsilon_\varphi)^2 + (\varepsilon_r - \varepsilon_z)^2 + (\varepsilon_\varphi - \varepsilon_z)^2 + \frac{3}{2}\left(\gamma_{r\varphi}^2 + \gamma_{rz}^2 + \gamma_{\varphi z}^2\right)\right]^{1/2}, \\ \varepsilon_r = \partial u/\partial r, \ \varepsilon_\varphi = u/r, \ \varepsilon_z = \partial w/\partial z, \\ \varepsilon_{rz} = (1/2)(\partial u/\partial z + \partial w/\partial r), \ u = u(r, z), \ w = w(r, z), \\ \frac{\partial^2 \varepsilon_r}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial r^2} = 2\frac{\partial^2 \varepsilon_{rz}}{\partial r\, \partial z}, \ \frac{\partial \varepsilon_\varphi}{\partial r} = \frac{\varepsilon_r - \varepsilon_\varphi}{r}. \end{gathered} \tag{1.2}$$

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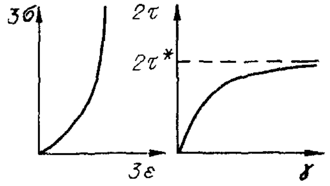

Fig. 1

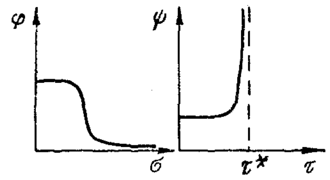

Fig. 2

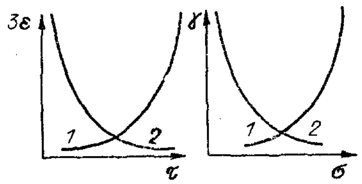

Fig. 3

Here $(2\psi)^{-1}$ is adjusted shear modulus (the understanding of shear modulus G is generalized); φ^{-1} is adjusted bulk strain modulus (the understanding of all-round compression modulus K is generalized); functions $\psi = \psi(|\sigma|, \tau)$ and $\varphi = \varphi(|\sigma|, \tau)$ are continuous, monotonic, positive, and they are even functions of their arguments.

For every elastic material there exists a function of six stress components $\Phi(\sigma_r, \sigma, \sigma_z, \ldots, \sigma_{\varphi z})$, exhibiting the properties

$$\varepsilon = (1/3)\partial\Phi/\partial\sigma,\ \gamma = \partial\Phi/\partial\tau \tag{1.3}$$

and the so-called specific additional work of deformation [2]. Taking account of (1.2) and (1.3) we have

$$\varphi = (1/3\sigma)\partial\Phi/\partial\sigma,\ \psi = (1/2\tau)\partial\Phi/\partial\tau.$$

Then the condition for existence of a potential may be written as

$$(\partial/\partial\tau)(3\sigma\varphi) = (\partial/\partial\sigma)(2\tau\psi). \tag{1.4}$$

Equations (1.1) and (1.2) are suitable for describing the stress–strained state of peaty, clay, sandy, and frozen soils.

It is well known that for soils under hydrostatic compression bulk modulus $K = E(1 - 2\nu)^{-1}$ normally increases and tends towards infinity, but with an increase in shear the strain modulus G decreases tending towards zero and $-\infty < \varphi_\sigma' \le 0$, and $\infty > \psi_\tau' \ge 0$ (Figs. 1 and 2). For soils there is also the property of dilation, i.e., a change in volume with shear. Soils of quite dense texture in shear start to increase in volume (the phenomenon of dilation) and loose soils decrease in volume (phenomenon of contraction). From the condition for existence of a potential (1.4) it follows that (Fig. 3): $\psi_\sigma' \ge 0$ with dilation (curve 1) and $\psi_\sigma' < 0$ with contraction (2) [2].

We consider the question of the independence of functions φ and ψ and the possibility of reversibility of the relationships $\varepsilon = \varphi\sigma$ and $\gamma = 2\psi\tau$, i.e., the possibility of presenting them in a form solved with respect to σ and τ: $\sigma = \eta(3\varepsilon, \gamma)\varepsilon$, $2\tau = \zeta(3\varepsilon, \gamma)\gamma$. For this it is necessary that it is Jacobian

$$\frac{D(\varphi, \psi)}{D(\sigma, \tau)} = \begin{vmatrix} \frac{\partial\varphi}{\partial\sigma} & \frac{\partial\varphi}{\partial\tau} \\ \frac{\partial\psi}{\partial\sigma} & \frac{\partial\psi}{\partial\tau} \end{vmatrix} = \varphi_\sigma'\psi_\tau' - \varphi_\tau'\psi_\sigma' \neq 0.$$

It is shown above that φ_σ' and ψ_τ' have different signs, but φ_τ' and ψ_σ' have the same signs, and consequently the Jacobian is negative and not equal to zero.

We assume that in the initial condition the material is under conditions of a uniform stress field. The initial stressed state for the material is given as

$$\sigma_r^0 = \sigma_\varphi^0 = -p, \sigma_z^0 = \tau_{rz}^0 = 0,\ \sigma^0 = -\frac{2}{3}p,\ \tau^0 = \frac{p}{\sqrt{3}}.$$

Then initial strains in accordance with (1.1) and (1.2) have the form

$$\varepsilon_\varphi^0 = \varepsilon_r^0 = -\frac{\psi^0 + 2\varphi^0}{3} p, \ \varepsilon_z^0 = -\frac{2(\varphi^0 - \psi^0)}{3} p, \ \varepsilon_{rz}^0 = 0.$$

Furthermore we assume that

$$\sigma_r = \sigma_r{}^0 + \sigma_r{}^*, \ \ldots, \ \varepsilon_r = \varepsilon_r{}^0 + \varepsilon_r{}^*, \ \ldots, \ u = u^0 + u^*, \ \ldots,$$
$$\sigma = \sigma^0 + \sigma^*, \ \tau = \tau^0 + \tau^*, \ \psi = \psi^0 + \psi^*, \ \varphi = \varphi^0 + \varphi^*,$$

and $\psi^* = \psi'^{0}_{|\sigma|}\sigma^* + \psi'^{0}_{\tau}\tau^*$, $\varphi^* = \varphi'^{0}_{|\sigma|}\sigma^* + \varphi'^{0}_{\tau}\tau^*$, where values with asterisks are small disturbances of the main stress, strain, and displacements fields. We linearize (1.1) and (1.2) with respect to their disturbances:

$$\partial\sigma_r^*/\partial r + \partial\tau_{rz}^*/\partial z + (\sigma_r^* - \sigma_z^*)/r = 0, \tag{1.5}$$
$$\partial\tau_{rz}^*/\partial r + \partial\sigma_z^*/\partial z + \tau_{rz}^*/r = 0,$$
$$3\varepsilon_r{}^* = \psi^0[\Sigma_{1r}\sigma_r{}^* + \Sigma_{1\varphi}\sigma_\varphi{}^* + \Sigma_{1z}\sigma_z{}^*],$$
$$3\varepsilon_\varphi{}^* = \psi^0[\Sigma_{1\varphi}\sigma_r{}^* + \Sigma_{1r}\sigma_\varphi{}^* + \Sigma_{3z}\sigma_z{}^*],$$
$$3\varepsilon_z^* = \psi^0\left[\Sigma_{3r}(\sigma_r^* + \sigma_\varphi^*) + \Sigma_{3z}\sigma_z^*\right], \varepsilon_{rz}^* = \psi^0\tau_{rz}^*,$$
$$\Sigma_{1r} = 2 + n + s, \ \Sigma_{1\varphi} = n - 1 + s, \ \Sigma_{1z} = n - 1 + r_1,$$
$$\Sigma_{3r} = n - 1 + q_1, \ \Sigma_{3z} = n + 2 + t, \ s = s_1 + 2s_2,$$
$$r_1 = s_3 + 2s_4, \ q_1 = 2(s_2 - s_1), \ t = 2(s_4 - s_3),$$
$$\sigma^* = (1/2)(\sigma_r{}^* + \sigma_\varphi{}^* + \sigma_z{}^*),$$
$$\tau = (1/2\sqrt{3})(2\sigma_z{}^* - \sigma_r{}^* - \sigma_\varphi{}^*),$$
$$\frac{\partial^2\varepsilon_r^*}{\partial z^2} + \frac{\partial^2\varepsilon_z^*}{\partial r^2} = 2\frac{\partial^2\varepsilon_{rz}^*}{\partial r\,\partial z}, \ \frac{\partial\varepsilon_\varphi^*}{\partial r} = \frac{\varepsilon_r^* - \varepsilon_\varphi^*}{r},$$
$$\varepsilon_r^* = \frac{\partial u^*}{\partial r}, \ \varepsilon_\varphi^* = \frac{u^*}{r}, \ \varepsilon_z^* = \frac{\partial w^*}{\partial z}, \ \varepsilon_{rz}^* = \frac{1}{2}\left(\frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial r}\right);$$
$$n = \varphi^0/\psi^0, \ s_1 = p\frac{\psi'^{0}_{|\sigma|}}{3\psi^0} - p\frac{\psi'^{0}_{\tau}}{2\sqrt{3}\,\psi^0}, \ s_2 = p\frac{\varphi'^{0}_{|\sigma|}}{3\psi^0} - p\frac{\psi'^{0}_{\tau}}{2\sqrt{3}\,\psi^0}, \tag{1.6}$$
$$s_3 = p\frac{\psi'^{0}_{|\sigma|}}{3\psi^0} + p\frac{\psi'^{0}_{\tau}}{\sqrt{3}\,\psi^0}, \ s_4 = p\frac{\varphi'^{0}_{|\sigma|}}{3\psi^0} + p\frac{\psi'^{0}_{\tau}}{\sqrt{3}\,\psi^0}.$$

Here dimensionless parameters n, s_1, s_2, s_3, s_4, are introduced which characterize the mechanical properties of the material of the medium in question and taking account of the prestressing regime.

From (1.5) we obtain equations for determining additional stresses:

$$\sigma_r^* = (\psi^0\Delta)^{-1}\left[\Sigma_1\frac{\partial u^*}{\partial r} + \Sigma_2\frac{u^*}{r} + \Sigma_3\frac{\partial w^*}{\partial z}\right], \sigma_\varphi^* = (\psi^0\Delta)^{-1}\left[\Sigma_2\frac{\partial u^*}{\partial r} + \Sigma_1\frac{u^*}{r} + \Sigma_3\frac{\partial w^*}{\partial z}\right], \tag{1.7}$$
$$\sigma_z^* = (\psi^0\Delta)^{-1}\left[\Sigma_4\left(\frac{\partial u^*}{\partial r} + \frac{u^*}{r}\right) + \Sigma_5\frac{\partial w^*}{\partial z}\right], \ \tau_{rz}^* = (2\psi^0)^{-1}\left[\frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial r}\right],$$
$$\Delta = (n + 2 + t)(1 + 2n + 2s) - 2(n - 1 + r_1)(n - 1 + q_1),$$
$$\Sigma_1 = \Sigma_{1r}\Sigma_{3z} - \Sigma_{3r}\Sigma_{1z}, \ \Sigma_2 = \Sigma_{3r}\Sigma_{1z} - \Sigma_{1\varphi}\Sigma_{3z},$$
$$\Sigma_3 = -3\Sigma_{1z}, \ \Sigma_4 = -3\Sigma_{3r}, \ \Sigma_5 = 3(\Sigma_{1r} + \Sigma_{1\varphi}).$$

Additional stresses in form (1.7) are placed in the equilibrium equation from (1.5), as a result of which we arrive at a set of equations with respect to additional displacements of the medium:

$$\left[L_1\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) + L_2\frac{\partial^2}{\partial z^2}\right]u^* + L_3\frac{\partial^2}{\partial r\,\partial z}w^* = 0, \tag{1.8}$$
$$L_4\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)u^* + \left[L_2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + L_5\frac{\partial^2}{\partial z^2}\right]w^* = 0,$$

where $L_i = L_i(n, s_1, s_2, s_3, s_4)$ (i = 1, 2, 3, 4) are functions of prestressing parameters (1.6) and $L_1 = 2\Sigma_1$, $L_2 = \Delta$, $L_3 = \Delta + 2\Sigma_3$, $L_4 = \Delta + 2\Sigma_4$, $L_5 = 2\Sigma_5$.

Conditions for ellipticity of the set of Eqs. (1.8) have the following form:

a) the case of complex conjugate roots of the determinant of system (1.8)

$$B^2 - 4AC < 0,\ \mu_1 = c + id,\ \mu_2 = \bar{\mu} = c - id, \tag{1.9}$$
$$\mu_3 = -\mu_1,\ \mu_4 = -\mu_2,$$
$$c = \frac{1}{\sqrt{2}}\left[-\frac{B}{2A} + \left(\frac{C}{A}\right)^{1/2}\right]^{1/2},\ d = \frac{1}{\sqrt{2}}\left[\frac{B}{2A} + \left(\frac{C}{A}\right)^{1/2}\right]^{1/2};$$

b) cases of real roots of the determinant of system (1.8)

$$B^2 \geqslant 4AC,\ A > 0,\ B < 0,\ C > 0;$$
$$B^2 \geqslant 4AC,\ A < 0,\ B > 0,\ C < 0;$$
$$\mu_1 = \left[\frac{-B + (B^2 - 4AC)^{1/2}}{2A}\right]^{1/2},\ \mu_2 = \left[\frac{-B - (B^2 - 4AC)^{1/2}}{2A}\right]^{1/2},$$
$$\mu_3 = -\mu_1,\ \mu_4 = -\mu_2$$
$$(A = L_2L_5,\ B = L_3L_4 - L_1L_5 - L_2^2,\ C = L_1L_2).$$

As an example we write conditions for ellipticity for a particular case of functions φ and ψ, and in fact when $\psi = \psi(\tau)$, but $\varphi = \text{const}$. In this way $s_2 = s_4 \equiv 0$, $n \not\equiv 0$, $s_1 = s \not\equiv 0$, $s_3 = -2s$, then system (1.8) is elliptical if

a)
$$\begin{aligned} &n < -2, && 0 < s < -(n+2); \\ &-2 < n < -0{,}5, && -(n+2) < s < 0; \\ &n > -0{,}5, && s < -(n+2),\ s > 0; \end{aligned} \tag{1.10}$$

b)
$$n < -2,\ -0{,}5 < s < 0,\ -(n+2) < s < -(1+2n)/2;$$
$$-2 \leqslant n < -1.5,\ -0.5 < s < -(n+2),\ 0 \leqslant s < -(1+2n)/2;$$
$$-1{,}5 \leqslant n \leqslant -0.75,\ -0.666 \leqslant n < -0.5,\ 0 < s < -(1+2n)/2;$$
$$n > 0,\ -0.5 < s < 0;$$
$$n = -2,\ -0.5 < s < -(1+2n)/2;$$
$$n \leqslant -1.5,\ n > 0,\ -(1+2n)/(2+3n) < s < -0.5;$$
$$-1.5 < n < -1,\ -(1+2n)/(2+3n) < s \leqslant -(n+2);$$
$$-0.5 < n < 0,\ -(1+2n)/2 < s < 0.$$

As follows from (1.10), critical values of quantities n and s satisfy the equations

$$s = -(n+2),\ n \leqslant -1.5,\ n \geqslant 1;$$
$$s = 0,\ n < -2,\ n \geqslant -1.5;$$
$$s = -(1+2n)/2,\ s = -(1+2n)/(2+3n),$$
$$s = -0.5 \quad (-\infty < n < \infty).$$

In Fig. 4 the region of ellipticity for the form of functions φ and ψ in question is hatched. From a mechanical point of view cases of loss of ellipticity for set of Eqs. (1.8) may be treated as cases of loss of internal stability of the medium as a result of prestressing [3, 4].

2. Now we consider the contact problem for a prestressed physically nonlinear layer. Let a layer which occupies region $(-\infty < r < \infty,\ 0 \leq \varphi \leq 2\pi,\ -h \leq z \leq 0)$ and prepared of

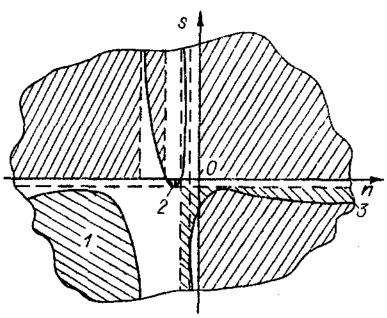

Fig. 4

material obeying relationships (1.2) lie without friction on a rigid base. At the boundary of the layer z = 0 force P is impressed by a rigid stamp circular in plan. We assume that frictional forces in the region of contact of the stamp and the layer are small and they may be ignored, and the radius of the contact region does not depend on the magnitude of the applied force. We shall consider action of the stamp on the layer as a small disturbance of the main stress field (1.2). We write boundary conditions for the problem formulated:

$$\begin{gathered}\tau_{rz}^*(r,-h)=0,\ w^*(r,-h)=0,\ \tau_{rz}^*(r,0)=0\quad (r<\infty);\\ \sigma_z^*(r,0)=0\ (r>a);\quad w^*(r,0)=-[\delta-f(r)]\ (r\leqslant a).\end{gathered}\tag{2.1}$$

For them it is necessary to add the requirement of extinction of stresses (additional in relation to the initial stress field) at infinity. Here δ is forward displacement of the stamp along axis z; f(r) is the shape of the stamp base.

In order to construct a solution of the problem posed we take an integral Hankel–Fourier transform with respect to variable r of equations of set (1.8) in the region of its ellipticity (1.9), i.e., we shall find disturbances of displacements u^* and w^* in the form

$$u^*(r,z)=\int_0^\infty U(\alpha,z)\alpha J_1(\alpha r)\,d\alpha,\ w^*(r,z)=\int_0^\infty W(\alpha,z)\alpha J_0(\alpha r)\,d\alpha.\tag{2.2}$$

In this way (1.8) for the Hankel–Fourier transform U and W takes the form

$$\begin{gathered}L_2U''(\alpha,z)-\alpha^2L_1U(\alpha,z)-\alpha L_3W'(\alpha,z)=0,\\ L_5W''(\alpha,z)-\alpha^2L_2W(\alpha,z)+\alpha L_4U'(\alpha,z)=0.\end{gathered}\tag{2.3}$$

Solution of system (2.3) is given by the equations

$$\begin{gathered}U(\alpha,z)=\frac{L_3\mu_1}{L_2\mu_1^2-L_1}[C_2(\alpha)\operatorname{sh}\alpha\mu_1z+C_1(\alpha)\operatorname{ch}\alpha\mu_1z]+\\ +\frac{L_3\mu_2}{L_2\mu_2^2-L_1}[C_4(\alpha)\operatorname{sh}\alpha\mu_2z+C_3(\alpha)\operatorname{ch}\alpha\mu_2z],\\ W(\alpha,z)=C_1(\alpha)\operatorname{sh}\alpha\mu_1z+C_2(\alpha)\operatorname{ch}\alpha\mu_1z+C_3(\alpha)\operatorname{sh}\alpha\mu_2z+C_4(\alpha)\operatorname{ch}\alpha\mu_2z\end{gathered}\tag{2.4}$$

($C_i(\alpha)$ (i = 1, 2, 3, 4) are unknown functions subject to determination from the boundary conditions for the problem).

Furthermore, we introduce into consideration a function for distribution of contact pressure q(r):

$$\sigma_z^*(r,0)=-q(r)\ (r\leqslant a)\tag{2.5}$$

and assuming that it is temporarily unknown we subject the first three boundary conditions of (2.1) together with (2.5) to integral Hankel–Fourier transform (2.2). Then by satisfying transformation boundary conditions by means of (2.4) and considering that additional stresses with $r\to\infty$ disappear in view of the properties of Hankel integrals, we find that

$$\begin{gathered}C_2(\alpha)=C_1(\alpha)\operatorname{th}\alpha\mu_1h,\\ C_1(\alpha)=\frac{[(L_3-L_2)\mu_2^2+L_1](\mu_1^2-\mu_2^2)L_1}{(L_2\mu_2^2-L_1)^2(L_2\mu_1^2-L_1)}\operatorname{ch}\alpha\mu_1h\operatorname{sh}\alpha\mu_2h\frac{\psi^0\Delta}{\Delta_1(\alpha)}\frac{Q(\alpha)}{\alpha},\\ C_3(\alpha)=\frac{[(L_3-L_2)(\mu_1^2+L_1)](\mu_1^2-\mu_2^2)L_1}{(L_2\mu_2^2-L_1)(L_2\mu_2^2-L_1)^2}\operatorname{sh}\alpha\mu_1h\operatorname{ch}\alpha\mu_2h\frac{\psi^0\Delta}{\Delta_1(\alpha)}\frac{Q(\alpha)}{\alpha},\\ C_4(\alpha)=C_3(\alpha)\operatorname{th}\alpha\mu_2h,\\ \Delta_1(\alpha)=\frac{(\mu_1^2-\mu_2^2)L_1}{(L_2\mu_1^2-L_1)^2(L_2\mu_2^2-L_1)^2}\{\mu_2[(L_3-L_2)\mu_1^2+L_1]\times\\ \times[L_3\Sigma_4+\Sigma_5(L_2\mu_2^2-L_1)]\operatorname{sh}\alpha\mu_1h\operatorname{ch}\alpha\mu_2h-\mu_1[(L_3-L_2)\mu_2+L_1]\times\\ \times[L_3\Sigma_4+\Sigma_5(L_2\mu_1^2-L_1)]\operatorname{sh}\alpha\mu_2h\operatorname{ch}\alpha\mu_1h\}.\end{gathered}$$

Here $Q(\alpha)=\int_0^\infty q(\rho)\rho J_0(\alpha\rho)\,d\rho$ is Hankel–Fourier transform for functions of the distribution of contact pressure. Now it is possible to write an expression for determining impression of the stamp

$$w^*(r, z) = \frac{L_1(\mu_1^2 - \mu_2^2)}{(L_2\mu_2^2 - L_1)^2 (L_2\mu_1^2 - L_1)^2} \psi^0 \Delta \int_0^a q(\rho)\rho d\rho \times$$
$$\times \int_0^\infty \Delta_1^{-1}(\alpha) \{[(L_3 - L_2)\mu_2^2 + L_1] \operatorname{sh} \alpha\mu_1(z + h) \operatorname{sh} \alpha\mu_2 h -$$
$$- [(L_3 - L_2)\mu_1^2 + L_2] \operatorname{sh} \alpha\mu_2(z + h) \operatorname{sh} \alpha\mu_1 h\} J_0(\alpha\rho) J_0(\alpha r)\, d\alpha.$$

By satisfying the last boundary condition of (2.1) we reduce the contact problem in question to a first order integral equation with respect to function $q(\rho)$:

$$\int_0^1 q(\rho)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho = \lambda\theta[\delta - f(r)] \qquad (r < 1). \tag{2.6}$$

The kernel of integral Eq. (2.6) has the form

$$k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) = \int_0^\infty L(\alpha) J_0\left(\frac{\alpha\rho}{\lambda}\right) J_0\left(\frac{\alpha r}{\lambda}\right) d\alpha, \tag{2.7}$$

where

$$L(\alpha) = \frac{\mu_1 b_4 - b_3\mu_2}{\mu_1 b_4 \operatorname{cth} \alpha\mu_1 - \mu_2 b_3 \operatorname{cth} \alpha\mu_2}, \quad \theta = \frac{\mu_1 b_4 - \mu_2 b_3}{\Delta L_1 L_3 (\mu_1^2 - \mu_2^2)}, \tag{2.8}$$
$$b_3 = [(L_3 - L_2)\mu_1^2 + L_1][\Sigma_4 L_3 + \Sigma_5(L_2\mu_2^2 - L_1)],$$
$$b_4 = [(L_3 - L_2)\mu_2^2 + L_1][\Sigma_4 L_3 + \Sigma_5(L_2\mu_1^2 - L_1)].$$

In relationships (2.6)-(2.8) λ, α, ρ, r, δ, f are dimensionless values, and $\lambda = ha^{-1}$ characterizes the relative thickness of the layer. It should be noted that with $\lambda \to \infty$ the integral equation for a prestressed physically nonlinear layer (2.6) is converted into an integral equation for the corresponding contact problem for a prestressed physically nonlinear half-space

$$\int_0^1 q(\rho)\rho d\rho \int_0^\infty J_0(\alpha\rho) J_0(\alpha r)\, d\alpha = \theta[\delta - f(r)] \quad (0 < r < 1).$$

An important characteristic of the problem for the half-space is parameter θ which in this case may be called the contact stiffness. Provided below are the results of studying values of contact stiffness for a particular case of functions φ and ψ, and in fact $\psi = \psi(\tau)$, φ = const:

$$\theta = \frac{2(n + 2 + s)}{[(1 + 2n) + (2 + 3n)s][1 + 2n - s + \sqrt{[1 + 2n + (2 + 3n)s](1 + 2n + 2s)}]}.$$

In the region for the change in parameters n and s where there is ellipticity of equations of set (1.8), θ takes real values, but it cannot be both positive and negative. Outside this region θ becomes complex. The region for negative values of contact stiffness, which are labelled 1-3 in Fig. 4, should be excluded from consideration because a negative value of θ contradicts the physical sense of the problem. It is interesting to note that the given regions are bounded by contours with which on passing through there is surface loss or deformability of the medium ($\theta = \infty$), or it is stable ($\theta = 0$). Contact stiffness reverts to infinity at the contours

$$s = -(1 + 2n)(2 + 3n)^{-1} \quad (n < -1,\ n \geqslant 0);$$
$$n = 0 \ (s \leqslant -2); \quad n = -0.5 \ (|s| < \infty)$$

and it takes a zero value with

$$s = -(n + 2) \ (|n| < \infty);$$
$$s = -0.5 \ (n = -0.5); \quad s = 0 \ (n = -2).$$

These processes are accompanied a simultaneous disruption of internal stability for the material of the medium as a result of prestressing. Given in Fig. 5 are curves for the dependence of θ on material mechanical properties and prior stressing.

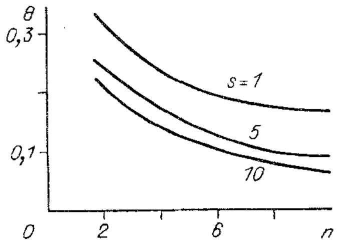

Fig. 5

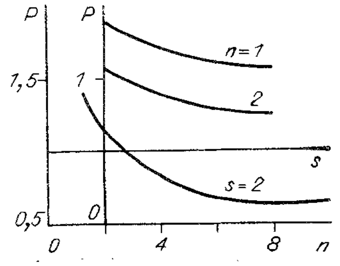

Fig. 6

3. We consider a comprehensive integral equation of the contact problem for a prestressed physically nonlinear layer (2.6). In external form it entirely conforms with the integral equation of the contact problem for a linear-elastic layer without initial stresses and is distinguished from it only in the form of the function for the symbol of the kernel $L(\alpha)$ and the value of dimensionless parameter θ. Analysis of expressions (2.8) showed that function $L(\alpha)$ exhibits all of the typical properties from [5]:

1) in the plane of complex variable $z = \alpha + i\beta$ function $L(z)z^{-1}$ is even and meromorphic, and with $\beta = 0$ it is real and regular;

2) $\lim\limits_{z\to 0} L(z)z^{-1} = D,\ D = \dfrac{\mu_1 b_4 - \mu_2 b_3}{b_4 - b_3} > 0;$

3) on a real axis with $|\alpha| \to \infty$ there is the estimate $L(\alpha) \approx 1 + O(e^{-\nu|\alpha|})$, $\nu = 2(\mu_1 + \mu_2) > 0$. Therefore in order to solve integral Eq. (2.6) it is possible to use asymptotic methods of "large and small λ" [5]. Results of studying (2.6) by means of these methods are presented in Fig. 6 in the form of a dependence for impression force $P = \int_0^1 q(r)dr$ with fixed sinking of the stamp (δ = const) and $\lambda = 3.1$ on mechanical properties of the medium ($\psi = \psi(\tau)$, φ = const) and prestressing conditions.

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